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A FAMILY OF ARC-TRANSITIVE GRAPHS OF GIRTH AT LEAST 5 ADMITTING A SUZUKI SIMPLE GROUP

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Abstract: We give a construction of a new family of Deza graphs of girth at least 5 that possess an arc-transitive group of automorphisms isomorphic to a Suzuki simple group $Sz(q)$. To study their combinatorical properties, we elaborate some group-theoretic arguments involving classical results on the groups of given type.

Keywords: Arc-transitive graph, Suzuki group, Graph of girth at least 5, Triangle-Free Graph, Deza graph.

1. Introduction

Studies of graphs with rich automorphism groups (namely, vertex-transitive, half-transitive or arc-transitive) satisfying various constraints for *girth*, i.e., the size of a smallest graph cycle, together with other structural characteristics have attained considerable attention of many researchers (for more background, e.g., see [1, 4, 5] and references therein). These graphs exhibit high symmetry possibly avoiding small cycles which in certain cases leads to important configurations having diverse applications in combinatorics and geometry.

In this paper, we give a construction of a new family of graphs of girth at least 5 that possess an arc-transitive (and even, arc-regular) group of automorphisms isomorphic to a Suzuki simple group $Sz(q)$. Since the family belongs to the class of Deza graphs, the construction also yields a new family of Deza graphs.

To introduce the construction and substantiate its properties, we will need the following description of the structure of Suzuki groups.

Proposition 1 (M. Suzuki [8]). *Let $G = Sz(q)$, where $q = 2^{2e+1} \geq 8$, S be an arbitrary Sylow 2-subgroup of G and $M := N_G(S)$ be its normalizer in G. Fix an involution $j \in G - M$. Then following statements hold.*

1. $|G| = q^2(q^2 + 1)(q - 1)$.
2. $|S| = q^2$ *and* $|Z(S)| = q$. *The group $Z(S)$ contains all the $q - 1$ involutions of S, and $H := M \cap M^j$ is a cyclic group of order $(q - 1)$, acting regularly (by conjugation) on both sets of non-identity elements of the groups $S/Z(S)$ and $Z(S)$.*
3. $M = S : H$.

4. *For every* $g \in G - M$ *there exists a unique element* $h \in H$ *and a unique pair of elements* $s_1, s_2 \in S$ *such that* $g = s_1 h j s_2$ *holds.*
5. $N_G(H) = \langle H, j \rangle \simeq D_{2(q-1)}$ *is a maximal subgroup in* G *of order* $2(q-1)$*. In particular,* $H = H^j$ *and* $h^j = h^{-1}$ *for any generator* h *of* H*.*
6. $G = \langle H, g \rangle$ *and* $M^g \cap M^j \cap M = 1$ *for any involution* $g \in G - M - M^j - N_G(H)$*.*

Throughout the paper, we keep notation for G, M, H, S and j from Proposition 1.

Definition 1. *Let* $g \in G - M - M^j - N_G(H)$ *be an involution. Let* $\Gamma := \Gamma(G, H, g)$ *denote the graph on the set* G/H *of the right* H*-cosets of* G*, in which the vertex* Hx *is adjacent to the vertex* Hy *if and only if* $xy^{-1} \in HgH$*.*

Recall that a *Deza graph with parameters* (n, k, a, b) is an undirected, simple graph with n vertices, regular of degree k, in which any pair of distinct vertices has either a or b common neighbours, where $a \le b$. (The rest terminology used in the paper is mostly standard and follows [2, 3, 8].) It is easy to see that arc-transitive graphs of girth at least 5 are just arc-transitive connected Deza graphs with $(a, b) = (0, 1)$ (e.g., see [2, Chapt. 1.2]). For a survey on Deza graphs and their known constructions, we refer the reader to [6].

Our aim is to study the next hypothesis.

Hypothesis 1. *For every* $q \ge 8$*, there is an involution*

$$g \in G - M - M^j - N_G(H)$$

such that Γ *is a Deza graph with parameters*

$$(n, k, a, b) = (q^2(q^2+1), q-1, 0, 1) \tag{1.1}$$

As a by-product of our construction, we also obtain a family of graphs of girth 4 admitting an arc-regular action of $Sz(q)$. We will show the following theorem.

Theorem 1. *For every* $q \ge 8$*, there is an involution*

$$g \in G - M - M^j - N_G(H)$$

such that either an equation $fgzg = gwgh$ *holds for some non-identity elements* $h, w, z, f \in H$ *and* Γ *has girth* 4*, or* Γ *is a Deza graph with girth at least* 5 *and parameters* (1.1)*. In each case,* Γ *admits* G *as an arc-transitive group of automorphisms with a trivial arc-stabilizer.*

2. Proof of Theorem 1

Lemma 1. Γ *is a simple, undirected, connected graph that admits the group* G *acting (by right multiplication) faithfully and transitively both on vertices and on arcs. The stabilizer of the vertex* $\{H\}$ *in* G *coincides with* H*.*

P r o o f follows by the well-known Sabidussi's theorem (e.g., see [3, p. 27]) and Proposition 1. □

Lemma 2. G *acts regularly on the arc set of* Γ*.*

P r o o f. By Lemma 1, G acts transitively on the arc set of Γ. Let us consider the stabilizer of the arc (H, Hg) in G. By definition of Γ, this is the set of all elements $x \in H$ such that $Hgx = Hg$, which implies that this stabilizer is $H \cap H^g$. Due to Proposition 1 and by the choice of the subgroup H and the element g, we have

$$H \cap H^g \subseteq M \cap M^j \cap M^g = 1,$$

i.e., $H \cap H^g = 1$. Moreover, the stabilizer of any other arc in G is conjugate to $H \cap H^g$, and thus trivial. Hence G acts regularly on the arc set of Γ. □

Lemma 3. *The following statements are true.*

1. Γ *has* $q^2(q^2+1)$ *vertices.*
2. Γ *is regular of degree* $q-1$.
3. *There is a constant* λ *such that each edge of* Γ *is contained in precisely* λ *triangles.*

P r o o f. As the vertex set $\mathrm{V}(\Gamma)$ of Γ is G/H, it follows from Proposition 1 and Lemma 1 that

$$|\mathrm{V}(\Gamma)| = |G : H| = \frac{q^2(q^2+1)(q-1)}{q-1} = q^2(q^2+1).$$

Due to transitivity of G on the vertices, it is sufficient to show that vertex H has valency $q-1$. Indeed, by Lemmas 1 and 2, H acts regularly on the neighbours of H. It means that elements of H are in one-to-one correspondence to neighbours of $\{H\}$, so $\deg\{H\} = |H| = q-1$.

Finally, Γ is edge-transitive, which implies that there is a constant λ such that every edge of Γ is contained in precisely λ triangles. □

Lemma 4. *The diameter of* Γ *is greater than two.*

P r o o f. Let d denote the diameter of Γ. Clearly, $d > 1$. We will show the impossibility of $d = 2$ by contradiction. Suppose $d = 2$. By Lemma 3, Γ is a regular graph of valency $q-1$, which implies there are at most $(q-1)(q-2)$ vertices at distance 2 from any given vertex. But then

$$|\mathrm{V}(\Gamma)| = q^2(q^2+1) \leq 1 + (q-1) + (q-1)(q-2),$$

which is impossible for all $q > 4$. □

In view of Lemmas 3 and 4, in order to check the Hypothesis 1 it remains to find an appropriate involution

$$g \in G - M - M^j - \langle H, j \rangle$$

(recall we set notation for G, S, M, H and j as in Proposition 1) such that $\lambda = 0$ and every two vertices at distance 2 of Γ have a constant number μ of common neighbors such that $\mu = 1$. This will be treated below in Lemmas 7 and 8.

Next we proceed with some technical preparations and definitions (most of these can be found in [8]). For any group A, we denote by $A^\#$ the set of non-identity elements of A. For each involution $i \in G - M$, we define the three mappings $\sigma_i, \tau_i : S^\# \to S$ and $\eta_i : S^\# \to M \cap M^i$ by the rule

$$isi = \sigma_i(s)\eta_i(s)i\tau_i(s) \quad \text{for all} \quad s \in S^\#.$$

In what follows, we denote $\sigma := \sigma_j$, $\tau := \tau_j$ and $\eta := \eta_j$ for the chosen involution $j \in G - M$. Since any two involutions in $G - M$ are conjugate in M, we have $g = j^m$ for some $m \in M$. Then

$$\sigma_g(s) = \sigma(s^{m^{-1}})^m, \quad \eta_g(s) = \eta(s^{m^{-1}})^m \quad \text{and} \quad \tau_g(s) = \tau(s^{m^{-1}})^m.$$

The element $m \in M = SH$ can be represented uniquely as $m = su$ for some $s \in S$ and $u \in H$. Hence $j^m = j^{su}$ and $H^m = M \cap M^{j s^u}$. Further, there is a unique pair of elements a and y in S which satisfy the structure identity

$$jaj = y^{-1}jy$$

of G with respect to j. Moreover, a is an involution in S such that $|aj| = 5$ and $y^2 = a$.

Lemma 5. *The functions σ, τ and η satisfy the following conditions:*

1) $\tau(\tau(s)) = s$,
2) $\sigma(s) = \tau(s^{-1})^{-1}$,
3) $\tau(h^{-1}sh) = h\tau(s)h^{-1}$,
4) $\sigma(h^{-1}sh) = h\sigma(s)h^{-1}$,
5) $\eta(h^{-1}sh) = h^2\eta(s)$,
6) $\tau(\sigma(s)) = \eta(s)^{-1}\tau(s)^{-1}\eta(s)$,
7) $\tau(cs) = \eta(s)\tau\big(\tau(c)\sigma(s)\big)\eta(s)^{-1}\tau(s)$,

where $s, c \in S$, $sc \neq 1$ and $h \in H$.

P r o o f was given in [8, Proposition 5]. □

Lemma 6. *For any $k \in H^{\#}$, the involution $g = j^{k^{-1}yk}$ is not contained in $M \cup M^j \cup N_G(H)$.*

P r o o f. By Proposition 1, the involutions in $M \cup M^j \cup N_G(H)$ have one of the following three forms: $a^h \in M = N_G(S)$, $a^{hj} \in M^j$ or $hj \in N_G(H) = \langle H, j\rangle$, where $h \in H$. Clearly, the involution $g = j^{k^{-1}yk}$ cannot be expressed in the first or the third form. By Lemma 5, $\eta(a^h) = h^2\eta(a) = h^2 \neq 1$ for all $h \in H^{\#}$, so g is not expressed in the second form either. □

Next we put $\pi(z) := y(y^{-1})^z$ for each $z \in H^{\#}$. We also put $[Hx]$ to denote the neighbourhood of a vertex Hx in Γ.

Lemma 7. *For each involution $g = j^{k^{-1}yk}$, where $k \in H^{\#}$, every edge of $\Gamma := \Gamma(G, H, g)$ is contained in $\lambda = 0$ triangles.*

P r o o f is given by contradiction. Take $k \in H^{\#}$ and put $g = j^{k^{-1}yk}$. Suppose $\lambda > 0$. Then the edge $\{H, Hg\}$ lies in a triangle $H \sim Hx \sim Hg$ for some vertex Hx. By definition it means that $x \in HgH$ and $xg \in HgH$, hence $x = h_1gh_2 = (w_1gw_2)g$ for some $h_i, w_i \in H$, where $i = 1, 2$. So

$$w_1^{-1}(h_1gh_2) = gw_2g.$$

Put $h := w_1^{-1}h_1$, $w := w_2$ and $z := h_2$. We have

$$hgz = hu^{-1}j^suz = u^{-1}j^suwu^{-1}j^su = u^{-1}j^swj^su.$$

Hence

$$\begin{aligned}(s^{-1})^{h^{-1}}hz^{-1}js^z = hs^{-1}z^{-1}js^z = hj^sz = j^swj^s = s^{-1}jsws^{-1}js \\ = s^{-1}w^{-1}js^ws^{-1}js = s^{-1}w^{-1}\sigma(\bar{s})\eta(\bar{s})j\tau(\bar{s})s = s^{-1}\sigma(\bar{s})^ww^{-1}\eta(\bar{s})j\tau(\bar{s})s,\end{aligned}$$

where $\bar{s} := s^w s^{-1} \neq 1$ (note that $w \neq 1$, as otherwise we would obtain $Hg = Hx$). Therefore,

$$\begin{aligned} \sigma(\bar{s}) &= (s(s^{-1})^{h^{-1}})^{w^{-1}}, \\ \tau(\bar{s}) &= s^z s^{-1} = (s(s^{-1})^z)^{-1} = (s(s^{-1})^{z^{-1}})^z, \\ \eta(\bar{s}) &= whz^{-1}. \end{aligned} \tag{2.1}$$

Recall that for every $f \in H$ there are exactly $q+1$ elements $s_0 \in S$ such that $\eta(s_0) = f$. All the $q-2$ of these elements which are not conjugate in H with either a, y, y^{-1} can be written uniquely in the form $s_0 = (\pi(k))^t$, where $t, k \in H$ and $k \neq 1$. Moreover, each of these $q-2$ elements belong to distinct H-orbits (with respect to the action of H by conjugation on S).

Also, for each $k \in H^\#$ there is a unique $b \in H^\#$ such that $k^{-1}a^k ak = a^b$ (since H acts regularly (by conjugation) on $Z(S)^\#$), and the mapping $\varphi : H^\# \to H$ by the rule $\varphi(k) = b$ is well-defined. The following holds for π (see [8, Section 12]):

$$\begin{aligned} \sigma(\pi(k)) &= y(y^{-1})^{\varphi(k)^{-1}k^2}, \\ \tau(\pi(k)) &= y^{\varphi(k)^{-1}}(y^{-1})^{k^{-1}}, \\ \eta(\pi(k)) &= \varphi(k)^2 k^{-2}. \end{aligned} \tag{2.2}$$

Now suppose there exists an element $k \in H^\#$ such that $s = y^k$ satisfies (2.1). Then

$$\bar{s} = s^w s^{-1} = (s(s^{-1})^{w^{-1}})^w = \pi(w^{-1})^{kw}.$$

And it follows from (2.1) and (2.2) that

$$\sigma(\pi(w^{-1})^{kw}) = \sigma(\pi(w^{-1}))^{(kw)^{-1}} = (y(y^{-1})^{\varphi(w^{-1})^{-1}w^{-2}})^{(kw)^{-1}} = (y(y^{-1})^{h^{-1}})^{kw^{-1}}, \tag{2.3}$$

$$\begin{aligned} \tau(\pi(w^{-1})^{kw}) &= \tau(\pi(w^{-1}))^{(kw)^{-1}} = (y^{\varphi(w^{-1})^{-1}}(y^{-1})^w)^{(kw)^{-1}} = \\ &= (y(y^{-1})^{\varphi(w^{-1})w})^{(\varphi(w^{-1})kw)^{-1}} = (y(y^{-1})^{z^{-1}})^{kz}, \end{aligned} \tag{2.4}$$

$$\eta(\pi(w^{-1})^{kw}) = k^2 w^2 \eta(\pi(w^{-1})) = k^2 w^4 \varphi(w^{-1})^2 = whz^{-1}. \tag{2.5}$$

Identities (2.3) and (2.5) imply that

$$h = \varphi(w^{-1})w^2 \quad \text{and} \quad k^2 w^2 \varphi(w^{-1})^2 = w^{-1}hz^{-1}.$$

It means

$$k^2 z = w^{-1}\varphi(w^{-1})^{-1}.$$

But then $k^2 \in C_H(\pi(h^{-1}))$ and consequently

$$k^2 z = w \quad \text{and} \quad w \neq \varphi(w^{-1})^{-1}.$$

Moreover, from the identity (2.4) we obtain that

$$(y(y^{-1})^{z^{-1}})^{kz} = (y(y^{-1})^{z^{-1}})^{k^{-1}w} = (y(y^{-1})^{\varphi(w^{-1})w})^{(\varphi(w^{-1})wk)^{-1}} = (y(y^{-1})^{w^{-1}})^{(w^{-1}k)^{-1}},$$

so $zw^{-1} \in C_H(y)$ and $z = w$. This leads to a contradiction since $k^2 \neq 1$. □

Hence for any involution

$$g = j^{k^{-1}yk}$$

with $k \in H^\#$, the parameter λ of $\Gamma(G, H, g)$ is zero. In the rest, we denote by φ the mapping defined in the proof of Lemma 7.

Lemma 8. *For each element $k \in H^{\#}$ and involution $g = j^{k^{-1}yk}$ either the parameter μ of $\Gamma = \Gamma(G,H,g)$ is well-defined and equals 1, or there are some elements $f, w, z, h \in H^{\#}$ that satisfy the equation $fgzg = gwgh$ which holds precisely if*

$$fz^3\varphi(z^{-1})^2 = h^{-1}w^3\varphi(w^{-1})^2,$$

$$\pi(f^{-1})^{k^2} \cdot \pi(\varphi(z)^{-1}z)^{f^{-1}} = \pi(\varphi(w)^{-1}w)$$

and

$$\pi(\varphi(z^{-1})^{-1}z^{-1}) = \pi(h)^{k^2}\pi(\varphi(w^{-1})^{-1}w^{-1})^{h}.$$

P r o o f. Suppose there is an element $h \in H^{\#}$ such that $\{Hg, Hgh\} \subseteq [H] \cap [Hx]$ for some Hx at distance 2 from H. By the definition of Γ, it means that $xg \in HgH$ and $x(gh)^{-1} \in HgH$. The latter inclusion takes place precisely if $xg = h_1gh_2$ and $xh^{-1}g = w_1gw_2$ for some $h_i, w_i \in H$, where $i = 1, 2$. So $x = h_1gh_2g = w_1gw_2gh$. Let us multiply this equality by w_1^{-1} on the left and denote $f := w_1^{-1}h_1$, $w := w_2$ and $z := h_2$. Thus we obtain

$$fgzg = fj^mzj^m = gwgh = j^mwj^mh = m^{-1}jw^{m^{-1}}jmh.$$

Now let us substitute $m = su$ to the previous equality:

$$fgzg = fu^{-1}j^suzu^{-1}j^su = fu^{-1}j^szj^su = u^{-1}j^suwu^{-1}j^suh = u^{-1}j^swj^suh,$$

so

$$fj^szj^s = j^swj^sh.$$

Furthermore, then

$$\begin{aligned}(s^{-1})^{f^{-1}}\sigma(\tilde{s})^{f^{-1}z}fz^{-1}\eta(\tilde{s})j\tau(\tilde{s})s &= (s^{-1})^{f^{-1}}fz^{-1}\sigma(\tilde{s})\eta(\tilde{s})j\tau(\tilde{s})s = (s^{-1})^{f^{-1}}fz^{-1}js^zs^{-1}js\\ &= fs^{-1}z^{-1}js^zs^{-1}js = fj^szj^s = j^swj^sh = s^{-1}jsws^{-1}jsh = s^{-1}w^{-1}js^ws^{-1}jsh\\ &= s^{-1}w^{-1}\sigma(\bar{s})\eta(\bar{s})j\tau(\bar{s})sh = s^{-1}\sigma(\bar{s})^ww^{-1}\eta(\bar{s})j\tau(\bar{s})sh = s^{-1}\sigma(\bar{s})^ww^{-1}\eta(\bar{s})h^{-1}j(\tau(\bar{s})s)^h,\end{aligned}$$

where $\tilde{s} := s^zs^{-1} \neq 1$ and $\bar{s} := s^ws^{-1} \neq 1$ since $z, w \neq 1$ (otherwise we would get $x \in H$).

It means that $\{Hg, Hgh\} \subseteq [H] \cap [Hx]$ for some $h \in H^{\#}$ if and only if there are elements $w, z \in H^{\#}$ and $f \in H$ such that

$$\begin{gathered}(s^{-1}\sigma(\tilde{s})^z)^{f^{-1}} = s^{-1}\sigma(\bar{s})^w,\\ \tau(\tilde{s})s = (\tau(\bar{s})s)^h,\\ fz^{-1}\eta(\tilde{s}) = w^{-1}h^{-1}\eta(\bar{s}).\end{gathered} \tag{2.6}$$

Let us fix an arbitrary element $k \in H^{\#}$ and assume that $s = y^k$. Then we have

$$\bar{s} = s^ws^{-1} = (s(s^{-1})^{w^{-1}})^w = \pi(w^{-1})^{kw},$$

and also

$$\tilde{s} = s^zs^{-1} = (s(s^{-1})^{z^{-1}})^z = \pi(z^{-1})^{kz}.$$

Suppose that for the chosen k, there exist $w, z, h \in H^{\#}$ and $f \in H$ such that the element s satisfies the identities (2.6). Therefore, by Lemma 5 and equalities from (2.2) it follows that

$$\begin{aligned}\sigma(\pi(w^{-1})^{kw}) &= \sigma(\pi(w^{-1}))^{(kw)^{-1}} = (y(y^{-1})^{\varphi(w^{-1})^{-1}w^{-2}})^{(kw)^{-1}},\\ \tau(\pi(w^{-1})^{kw}) &= \tau(\pi(w^{-1}))^{(kw)^{-1}} = (y(y^{-1})^{\varphi(w^{-1})w})^{(\varphi(w^{-1})kw)^{-1}},\\ \eta(\pi(w^{-1})^{kw}) &= k^2w^2\eta(\pi(w^{-1})) = k^2w^4\varphi(w^{-1})^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma(\pi(z^{-1})^{kz}) &= \sigma(\pi(z^{-1}))^{(kz)^{-1}} = (y(y^{-1})^{\varphi(z^{-1})^{-1}z^{-2}})^{(kz)^{-1}},\\ \tau(\pi(z^{-1})^{kz}) &= \tau(\pi(z^{-1}))^{(kz)^{-1}} = (y(y^{-1})^{\varphi(z^{-1})z})^{(\varphi(z^{-1})kz)^{-1}},\\ \eta(\pi(z^{-1})^{kz}) &= k^2z^2\eta(\pi(z^{-1})) = k^2z^4\varphi(z^{-1})^2.\end{aligned}$$

By substituting these equalities to (2.6), we obtain

$$\begin{gathered}\left((y^{-1})^k \cdot (y(y^{-1})^{\varphi(z^{-1})^{-1}z^{-2}})^{k^{-1}}\right)^{f^{-1}} = (s^{-1}\sigma(\tilde{s})^z)^{f^{-1}} = s^{-1}\sigma(\bar{s})^w\\ = (y^{-1})^k \cdot (y(y^{-1})^{\varphi(w^{-1})^{-1}w^{-2}})^{k^{-1}},\\ (y(y^{-1})^{\varphi(z^{-1})z})^{(\varphi(z^{-1})kz)^{-1}} \cdot y^k = \tau(\tilde{s})s = (\tau(\bar{s})s)^h = ((y(y^{-1})^{\varphi(w^{-1})w})^{(\varphi(w^{-1})kw)^{-1}} \cdot y^k)^h,\\ fz^{-1}k^2z^4\varphi(z^{-1})^2 = w^{-1}h^{-1}k^2w^4\varphi(w^{-1})^2.\end{gathered}\tag{2.7}$$

Now, it follows from the identity (2.7) for η that

$$fz^3\varphi(z^{-1})^2 = h^{-1}w^3\varphi(w^{-1})^2,\tag{2.8}$$

which yields

$$h = f^{-1}z^{-3}\varphi(z^{-1})^{-2}w^3\varphi(w^{-1})^2.$$

If $f = 1$ then (2.7) can be rewritten the following way:

$$\begin{gathered}(y^{-1})^{\varphi(z^{-1})^{-1}z^{-2}} = (y^{-1})^{\varphi(w^{-1})^{-1}w^{-2}},\\ (y(y^{-1})^{\varphi(z^{-1})z})^{(\varphi(z^{-1})z)^{-1}} \cdot y^{k^2} = (y(y^{-1})^{\varphi(w^{-1})w})^{(\varphi(w^{-1})w)^{-1}h} \cdot y^{k^2h},\\ z^3\varphi(z^{-1})^2 = h^{-1}w^3\varphi(w^{-1})^2.\end{gathered}\tag{2.9}$$

Since

$$\varphi(z^{-1})^{-1}z^{-2} \cdot (\varphi(w^{-1})^{-1}w^{-2})^{-1} \in C_H(y) = 1,$$

we get

$$h = (z^3\varphi(z^{-1})^2)^{-1}w^3\varphi(w^{-1})^2 = zw^{-1}.$$

Substituting this equality to (2.9) identity for τ:

$$(y(y^{-1})^{\varphi(z^{-1})z})^{(\varphi(z^{-1})z)^{-1}} \cdot y^{k^2} = (y(y^{-1})^{\varphi(w^{-1})w})^{(z\varphi(z^{-1}))^{-1}} \cdot y^{k^2h}.$$

Then conjugation by $\varphi(z^{-1})z$ yields

$$y(y^{-1})^{\varphi(z^{-1})z} \cdot y^{k^2\varphi(z^{-1})z} = y(y^{-1})^{\varphi(w^{-1})w} \cdot y^{k^2h\varphi(z^{-1})z}.$$

Cancelling y on the left we obtain that

$$(y^{-1})^{\varphi(z^{-1})z} \cdot y^{k^2\varphi(z^{-1})z} = (y^{-1})^{\varphi(w^{-1})w} \cdot y^{k^2h\varphi(z^{-1})z} = (y^{-1})^{\varphi(z^{-1})z^2w^{-1}} \cdot y^{k^2h\varphi(z^{-1})z}.$$

Furthermore, conjugation by $(\varphi(z^{-1})z)^{-1}$ gives

$$y^{-1}y^{k^2} = (y^{-1})^{zw^{-1}} \cdot y^{k^2h} = (y^{-1}y^{k^2})^{zw^{-1}},$$

so

$$zw^{-1} \in C_H(y^{-1}y^{k^2}) = 1,$$

which contradicts the assumption $h \neq 1$.

Hence $f \neq 1$. Then by (2.7) for σ we get

$$((y^{-1})^k y^{k^{-1}})^{f^{-1}} (y^{-1})^{\varphi(z^{-1})^{-1} z^{-2} k^{-1} f^{-1}} = ((y^{-1})^k y^{k^{-1}})^{f^{-1}} (y^{-1})^{h\varphi(z^{-1}) z k^{-1} w^{-3} \varphi(w^{-1})^{-2}}$$
$$= (y^{-1})^k y^{k^{-1}} (y^{-1})^{\varphi(w^{-1})^{-1} w^{-2} k^{-1}}.$$

Multiplying both sides on the left by y^k, we obtain

$$y^k (y^{-1})^{k f^{-1}} \cdot y^{k^{-1} f^{-1}} (y^{-1})^{\varphi(z^{-1})^{-1} z^{-2} k^{-1} f^{-1}} = y^{k^{-1}} (y^{-1})^{\varphi(w^{-1})^{-1} w^{-2} k^{-1}}.$$

This identity can be rewritten in the following form

$$\pi(f^{-1})^k \cdot y^{k^{-1} f^{-1}} (y^{-1})^{\varphi(z^{-1})^{-1} z^{-2} k^{-1} f^{-1}} = y^{k^{-1}} (y^{-1})^{\varphi(w^{-1})^{-1} w^{-2} k^{-1}}.$$

Now, conjugation by k gives

$$\pi(f^{-1})^{k^2} \cdot (y (y^{-1})^{\varphi(z^{-1})^{-1} z^{-2}})^{f^{-1}} = y (y^{-1})^{\varphi(w^{-1})^{-1} w^{-2}}.$$

If

$$\varphi(z^{-1})^{-1} z^{-2} = 1,$$

then

$$f^{-1} = \varphi(w^{-1})^{-1} w^{-2}$$

and $k^2 \in C_H(\pi(f^{-1})) = 1$, a contradiction. Similarly, if $\varphi(w^{-1})^{-1} w^{-2} = 1$, then $f^{-1} = \varphi(z^{-1}) z$ and, again, $k^2 \in C_H(\pi(f^{-1})) = 1$, a contradiction.

Thus we have

$$\pi(f^{-1})^{k^2} \cdot \pi(\varphi(z^{-1})^{-1} z^{-2})^{f^{-1}} = \pi(\varphi(w^{-1})^{-1} w^{-2}).$$

Note that $\varphi(v) = \varphi(v^{-1}) v^3$ regardless of the choice of $v \in H^{\#}$, therefore

$$\pi(f^{-1})^{k^2} \cdot \pi(\varphi(z)^{-1} z)^{f^{-1}} = \pi(\varphi(w)^{-1} w). \tag{2.10}$$

On the other hand, conjugation by k and right multiplication by $(y^{-1})^{k^2}$ of both sides of (2.7) for τ gives

$$(y (y^{-1})^{\varphi(z^{-1}) z})^{(\varphi(z^{-1}) z)^{-1}} = (y (y^{-1})^{\varphi(w^{-1}) w})^{(\varphi(w^{-1}) w)^{-1} h} \pi(h^{-1})^{h k^2}.$$

If $\varphi(z^{-1})^{-1} z^{-2} = 1$, then $h^{-1} = \varphi(w^{-1})^{-1} w^{-1}$ and $k^2 \in C_H(\pi(h^{-1})^h) = 1$, a contradiction. Using same technique, if $\varphi(w^{-1})^{-1} w^{-1} = 1$, then $h^{-1} = \varphi(z^{-1}) z$ and it also implies

$$k^2 \in C_H(\pi(h^{-1})^h) = 1,$$

a contradiction. So we have

$$\pi(\varphi(z^{-1}) z)^{(\varphi(z^{-1}) z)^{-1}} = \pi(\varphi(w^{-1}) w)^{(\varphi(w^{-1}) w)^{-1} h} \pi(h^{-1})^{h k^2}.$$

Since $\pi(v)^{v^{-1}} = \pi(v^{-1})^{-1}$ for all $v \in H^{\#}$, the latter identity yields

$$\pi(\varphi(z^{-1})^{-1} z^{-1})^{-1} = (\pi(\varphi(w^{-1})^{-1} w^{-1})^{-1})^h (\pi(h)^{-1})^{k^2} = (\pi(h)^{k^2} \pi(\varphi(w^{-1})^{-1} w^{-1})^h)^{-1}.$$

Hence

$$\pi(\varphi(z^{-1})^{-1} z^{-1}) = \pi(h)^{k^2} \pi(\varphi(w^{-1})^{-1} w^{-1})^h. \tag{2.11}$$

Now, if $Hgf_1, Hgf_2 \in [H] \cap [Hz_1]$ for some Hz_1 at distance 2 from H, then

$$Hg, Hgf_2 f_1^{-1} \in [H] \cap [Hz_1 f_1^{-1}]$$

and $Hz_1 f_1^{-1}$ is also at distance 2 from H. In view of transitivity of G on the vertices of Γ, by repeating the above argument for $h = f_2 f_1^{-1}$ and $x = z_1 f_1^{-1}$, we obtain the required statement. □

P r o o f of Theorem 1 follows from Lemmas 2, 3, 4, 6, 7, 8. □

Hypothesis 2. *Take $k \in H^{\#}$. Suppose that for the involution $g = j^{k^{-1}yk}$ there is a quadruple (f, w, z, h) of non-identity elements of H that satisfy the equations (2.8), (2.10) and (2.11). Then $f = z = w^{-1} = h^{-1}$ and there is no any such a quadruple satisfying these three equations for the involution $g_1 = j^{kyk^{-1}}$.*

3. Concluding remarks

The hypothesis 2 was checked to be true for small values of $q = 8, 32$ using computations in GAP [7, 9]. If it holds in general case, then for each q there are precisely $(q-1)/2$ suitable involutions g, which in view of Theorem 1 yield $(q-1)/2$ (possibly, non-isomorphic) arc-transitive Deza graphs $\Gamma = \Gamma(G, H, g)$ of girth at least 5.

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